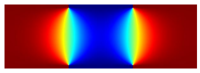
Liebmann technical documentation



Laplace equation 2D (ZR)
(Cylindrical coordinates).
relaxation scheme explained.
(5 - point star)

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2 Versions of this document

1. version 1 - 2023.11.03
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5. version 5 - 2024.05.18
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7. version 7 - 2024.05.24
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3 Solving Laplace equation using relaxation method

I tried to solve Laplace equation using mainly information from Pierre Grivet's book (Electron Optics) - [1].
There are few editions of this book (1965, 1972). Second edition (1972) contains explanation of relaxation method (page 38).

More generalized approaches has been drafted by James R. Nagel - [2]. `https://my.ece.utah.edu/~ece6340/LECTURES/Feb1/` (visited 2023-03-01).

Taylor expansion in cylindrical coordinates has been found on the Internet: [3].

There are also publications edited by Albert Septier: Focusing of Charged Particles [4] and Applied Charged Particle Optics (part A). [5].

I have also found some ideas in publication of D W O Heddle: Electrostatic Lens Systems [6] (especially using PC computers to solve electrostatic problems).

I have also found (brief) description of by - hand solving of Laplace equation by Bohdan Paszkowski - [7] (Polish edition). English translation of this book also exists - [8].

I would like to thank many people, who helped me with this challenge. Especially prof. dr hab. Mieczysław Jałochowski (supervisor of my master's thesis), who enabled me to use SIMION and MATLAB software while writing master's thesis about electron optical systems at University of Maria Curie - Skłodowska in Lublin in 2008. I would also thank to prof. Marcin Turek for fruitful discussion about numerical methods. What is more, my colleague Bartosz in 2012 had explained me general problems with software efficiency. So he had also contributed significantly to the idea of Liebmann software (especially using C language).

4 Explanation of symbols in calculations

- P_i - i-th mesh node
- V_i - value of electrostatic potential at node P_i. Unit - $[\mathrm{V}]$
- h - mesh step (for example h_x - mesh step in x direction). Unit - $[\mathrm{mm}]$
- $g_{i+/-}$ - gradient in direction i (for example $g_{1z-} = \frac{V_1 - V_{1z-}}{h_z}$. Unit - $\left[\frac{\mathrm{V}}{\mathrm{mm}}\right]$
- i_{row} - index of row in mesh. Values of $i_{row} = 1, 2, .., \mathrm{size\_row}$
- i_{col} - index of column in mesh. Values of $i_{col} = 1, 2, .., \mathrm{size\_col}$
- p - in book: - [1] $r = ph_r$, so for off - axis point we have: $p = (i_{row} - 1)$

5 Mesh ZR - type A (on axis)

$h_z \neq h_r$

gradient V outside a mesh exists

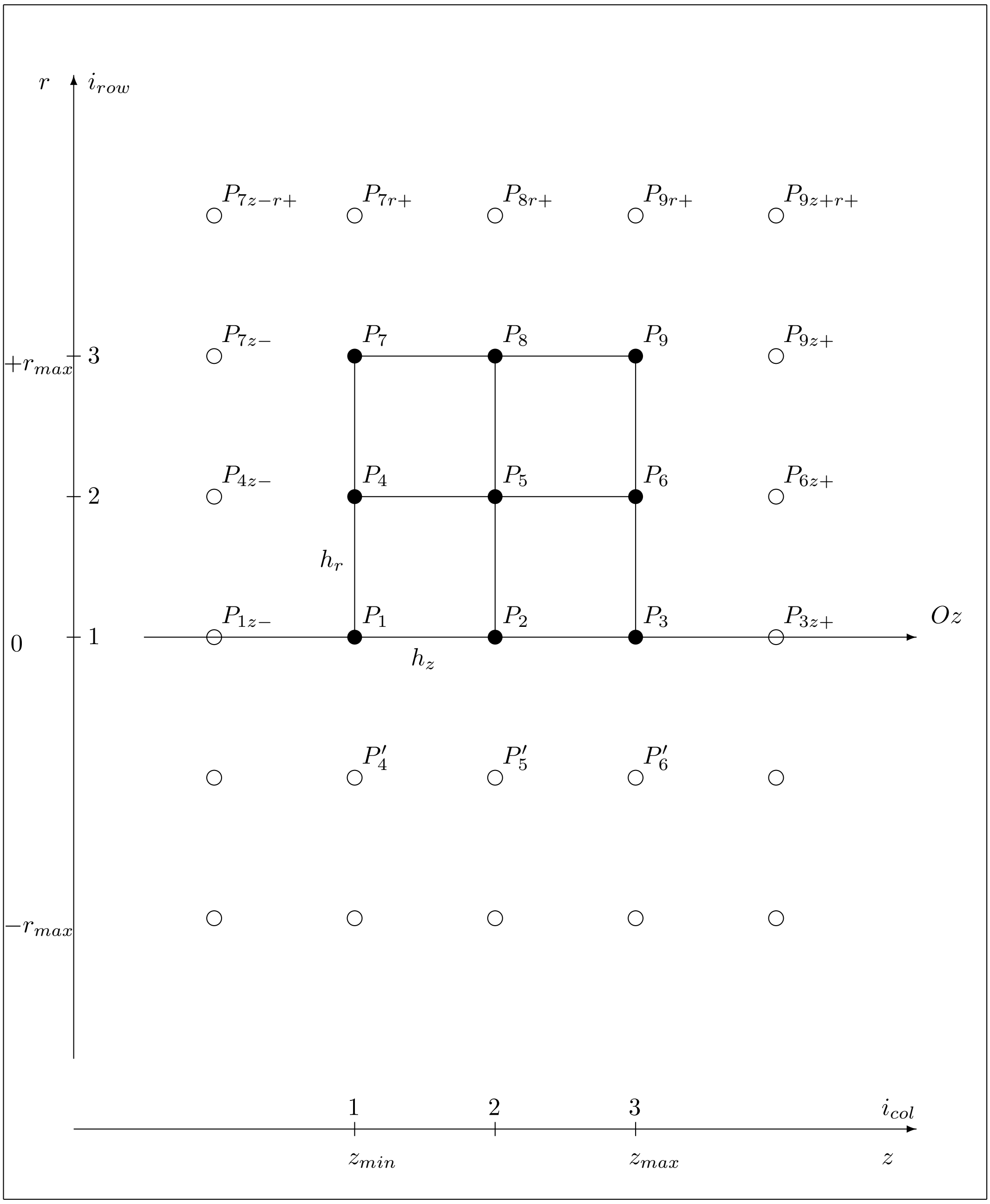

Figure 1: Mesh ZR type A

6 Mesh ZR - type B (on axis)

$h_z \neq h_r$

gradient V outside a mesh does not exist

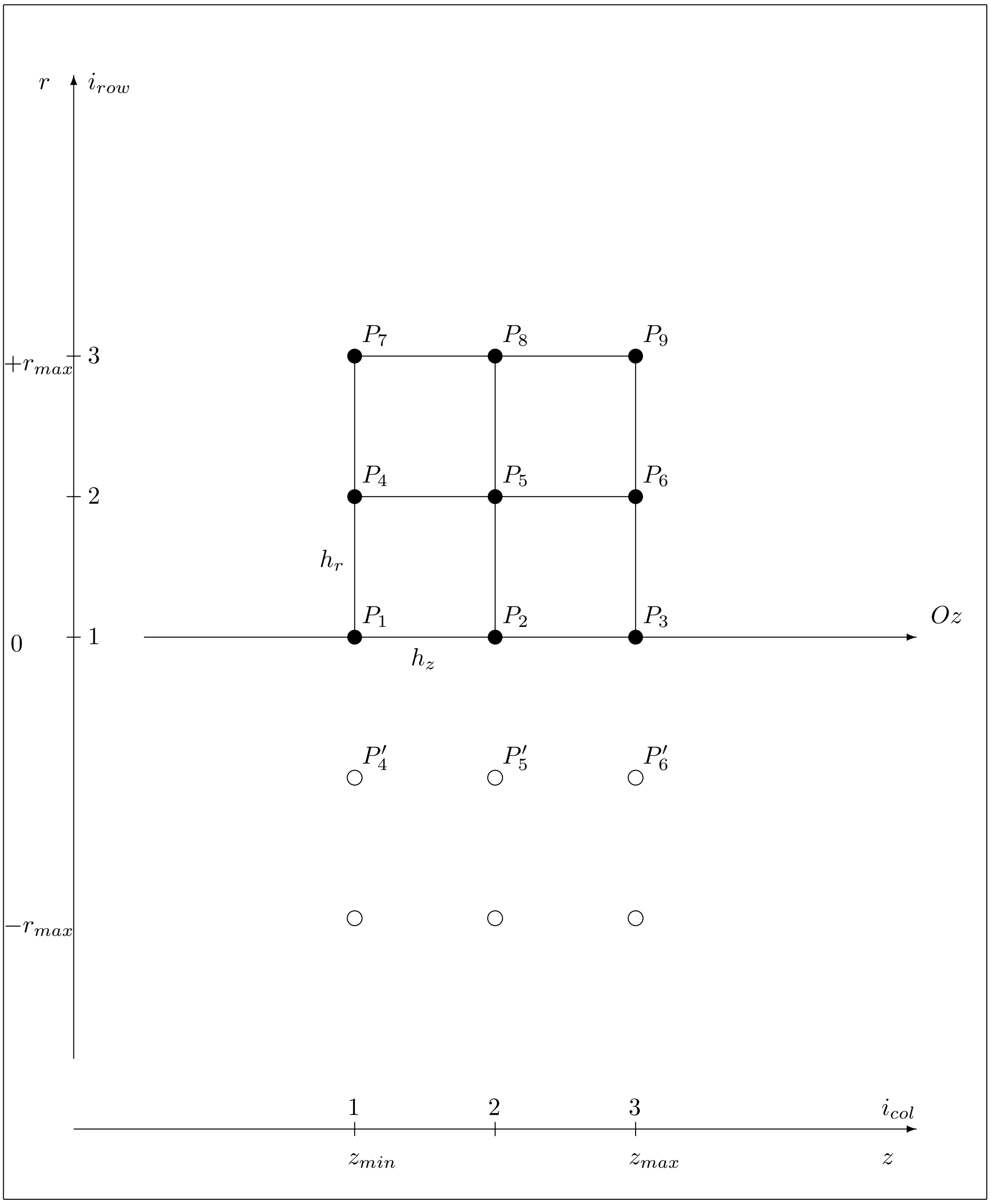

Figure 2: Mesh ZR type B

7 Mesh ZR - type C (on axis)

$h_z = h_r = h$
gradient V outside a mesh exists

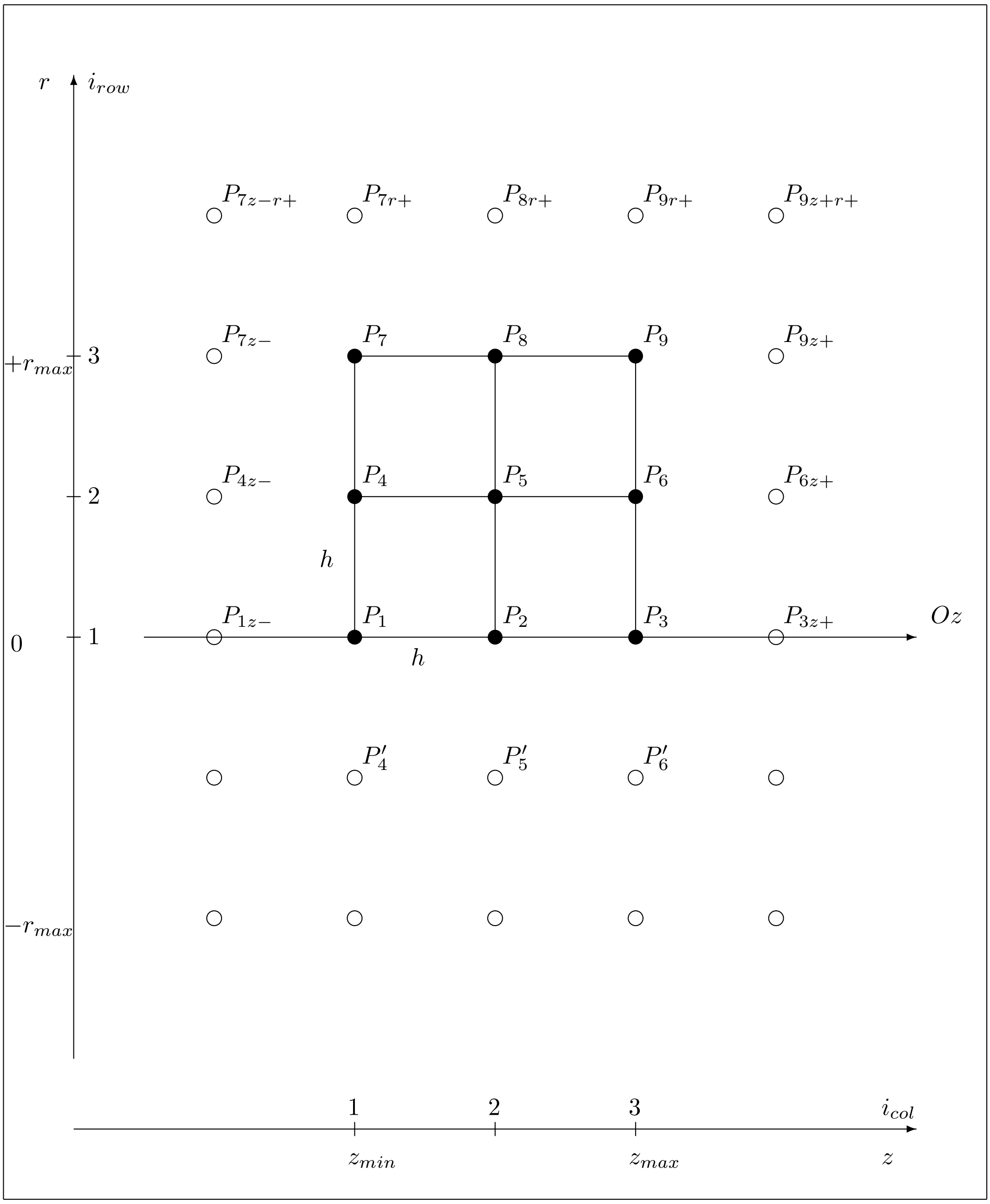

Figure 3: Mesh ZR type C

8 Mesh ZR - type D (on axis)

$h_z = h_r = h$
gradient V outside a mesh does not exist

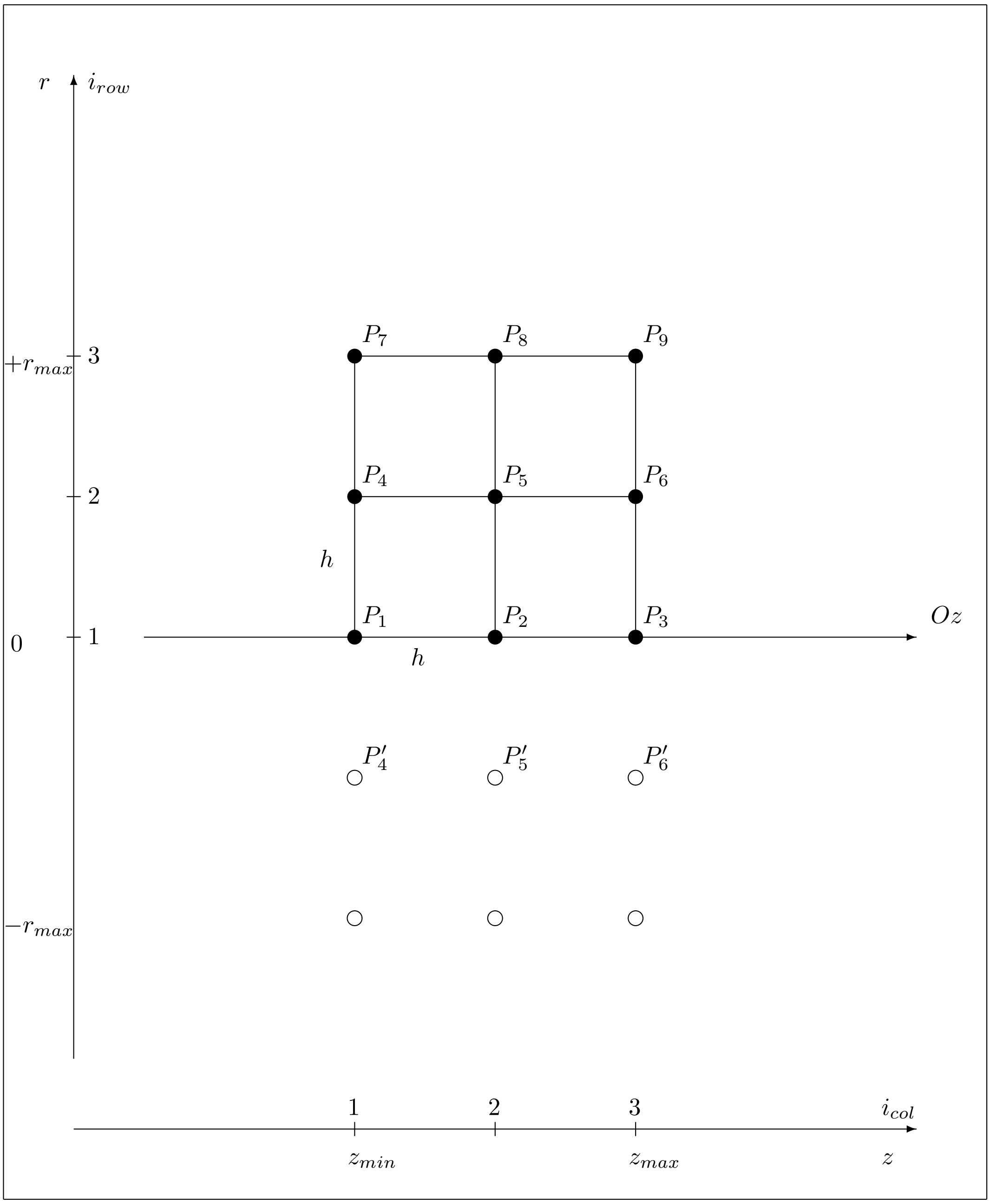

Figure 4: Mesh ZR type D

9 Example of A-type mesh in ANSI C (on axis)

Example of A- type mesh in ANSI C program. The mesh is represented by 2 dimensional array of double precision numbers. Rows and columns in mesh are numbered from 1 (this was my choice) instead of default 0 (as usual in C language). This choice nas pros and cons. Is is easier to calculate mesh size (size_row * size_col). Access to each node can be also more intuitive, but logic in each library function must contain this shift between node ordering styles.

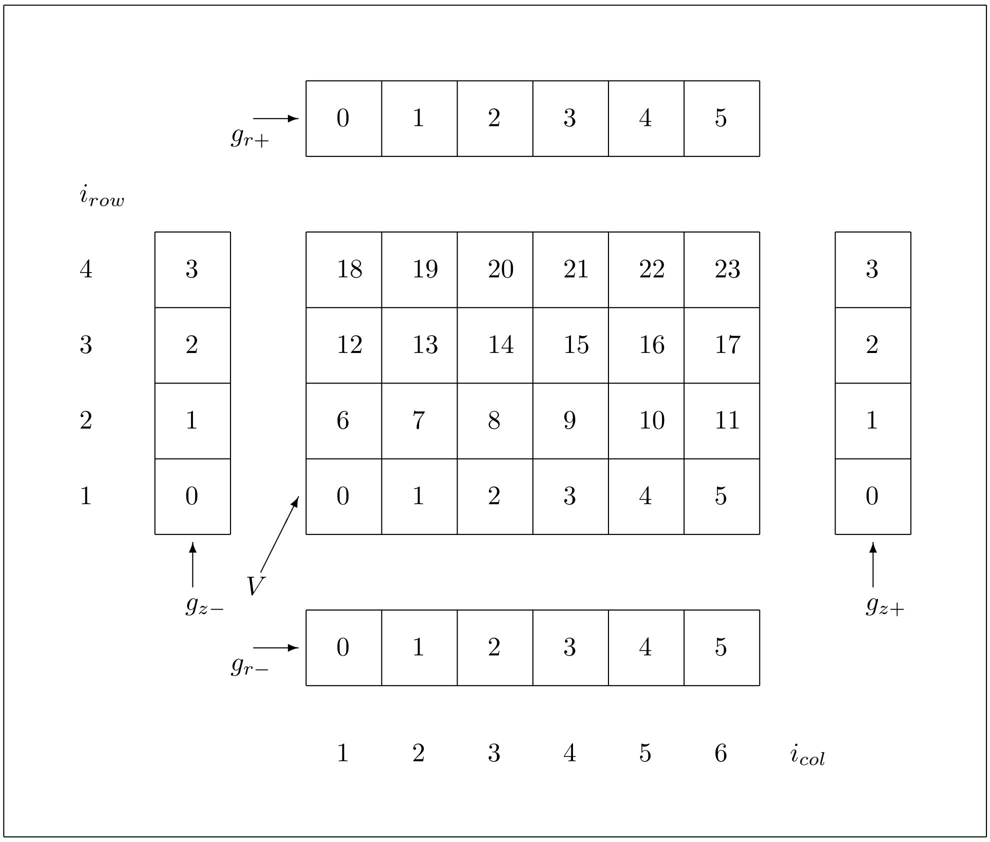

Figure 5: ANSI C - mesh XY type A

Note. This is more general example of „off-axis” mesh. If bottom egde of mesh lies on axis Oz, then gradient g_{r-} does not exist.

- $g_{z-} \equiv$ `double* ptr_gZ_minus`
- $g_{z+} \equiv$ `double* ptr_gZ_plus`
- $g_{r-} \equiv$ `double* ptr_gR_minus`
- $g_{r+} \equiv$ `double* ptr_gR_plus`

- $V \equiv$ `double* ptr_V`
- `unsigned int size_row == 4`
- `unsigned int size_col == 6`
- `unsigned int i_row == 1, 2, .., 4`
- `unsigned int i_col == 1,2, .., 6`
- `double h_z == 1.0 [mm]`
- `double h_r == 2.0 [mm]`

The following picture describes analogous version of `ptr_V` mesh, which can be dynamically allocated on heap by pointer metod. The mesh is represented by single block of memory. The numbers or rows and columns are also known, so each node can be also accessed by appropriate index (memory address).

The following picture describes analogous version of mesh, which can be easily dynamically allocated on heap by pointer metod. The mesh is represented by single block of memory. The numbers or rows and columns are also known, so each node can be also accessed by appropriate index (memory address).

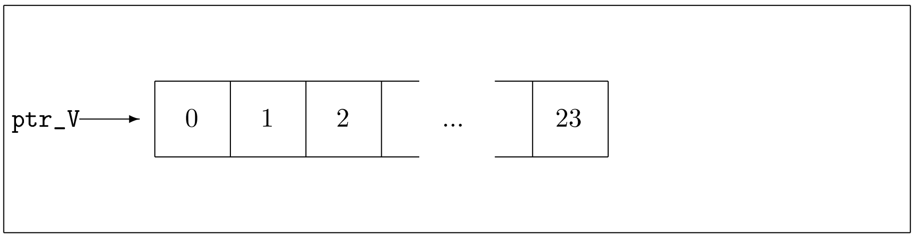

Figure 6: ANSI C - mesh ZR type D

Each mesh point has its unique index (let's say `icp` - (index of central point)), which can be determined, if we know indices of row and column (`i_row`, `i_col`).

$$\texttt{icp == (i\_row - 1) * size\_col + i\_col - 1} \tag{9.1}$$

For example for each point of a mesh indices of row and column have values:

$$\begin{aligned}&\texttt{i\_row == 1, 2, ..\ \ , size\_row}\\&\texttt{i\_col == 1, 2, ..\ \ , size\_col}\end{aligned} \tag{9.2}$$

Also for any relaxation formula for off - axis case the p symbol appears. This symbol is connected with r cylindrical coordinate of given node:

$$r = ph_r \tag{9.3}$$

so:

$$\texttt{p == (i\_row - 1)} \tag{9.4}$$

10 Example of B-type mesh in ANSI C (on axis)

Example of B- type mesh in ANSI C program. The mesh is analogous to A - type mesh. There are no electric field gradients on mesh borders.

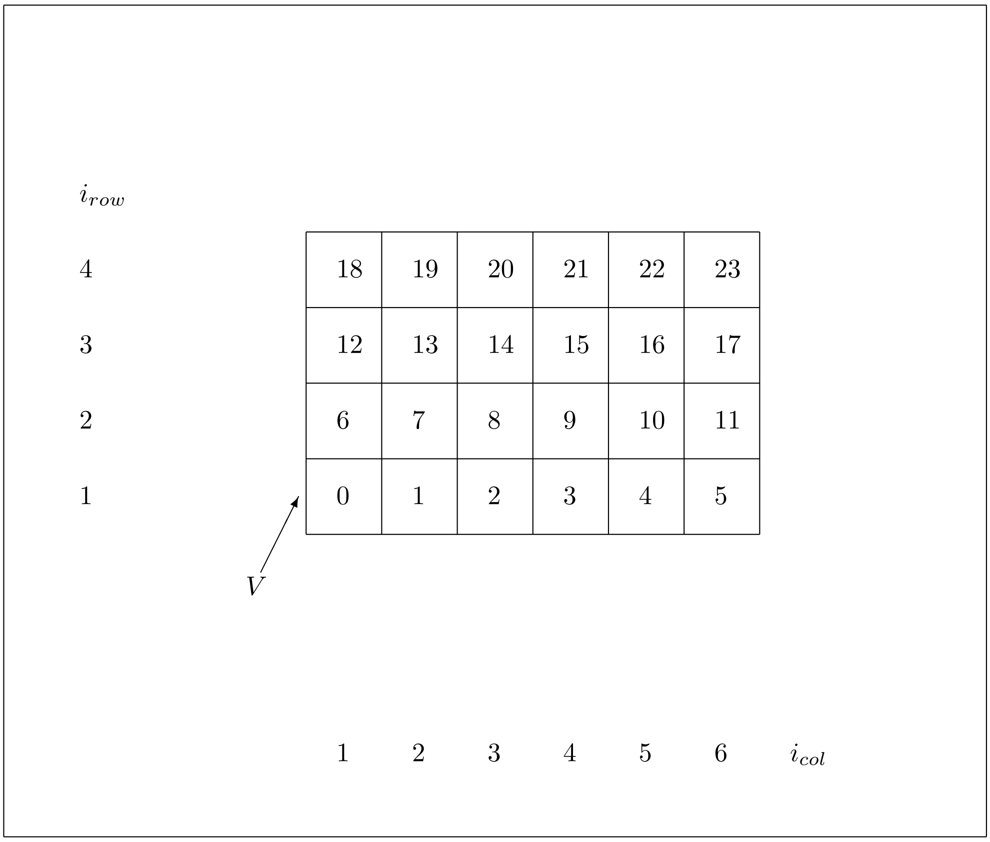

Figure 7: ANSI C - mesh XY type B

- $V \equiv$ `double* ptr_V`
- `unsigned int size_row == 4`
- `unsigned int size_col == 6`
- `unsigned int i_row == 1, 2, .., 4`
- `unsigned int i_col == 1,2, .., 6`
- `double h_z == 1.0 [mm]`
- `double h_r == 2.0 [mm]`

11 Example of C-type mesh in ANSI C (on axis)

Example of C- type mesh in ANSI C program. The mesh is analogous to A - type mesh. Just mesh mesh step $h_x = h_y = h$.

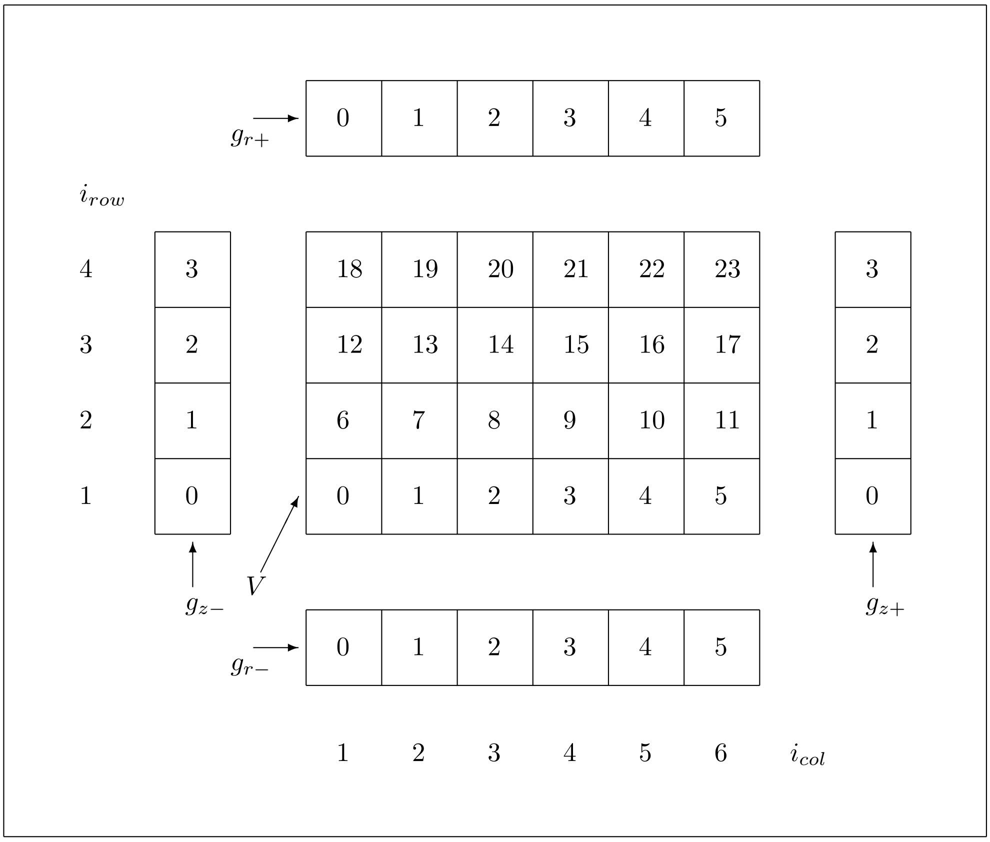

Figure 8: ANSI C - mesh XY type C

Note. This is more general example of „off-axis” mesh. If bottom egde of mesh lies on axis Oz, then gradient g_{r-} does not exist.

- $g_{z-} \equiv$ `double* ptr_gZ_minus`
- $g_{z+} \equiv$ `double* ptr_gZ_plus`
- $g_{r-} \equiv$ `double* ptr_gR_minus`
- $g_{r+} \equiv$ `double* ptr_gR_plus`
- $V \equiv$ `double* ptr_V`
- `unsigned int size_row == 4`

- unsigned int size_col == 6
- unsigned int i_row == 1, 2, .., 4
- unsigned int i_col == 1,2, .., 6
- double h == 1.0 [mm]

12 Example of D-type mesh in ANSI C (on axis)

Example of D- type mesh in ANSI C program. The mesh is analogous to B - type mesh. Just $h_x = h_y = h$.

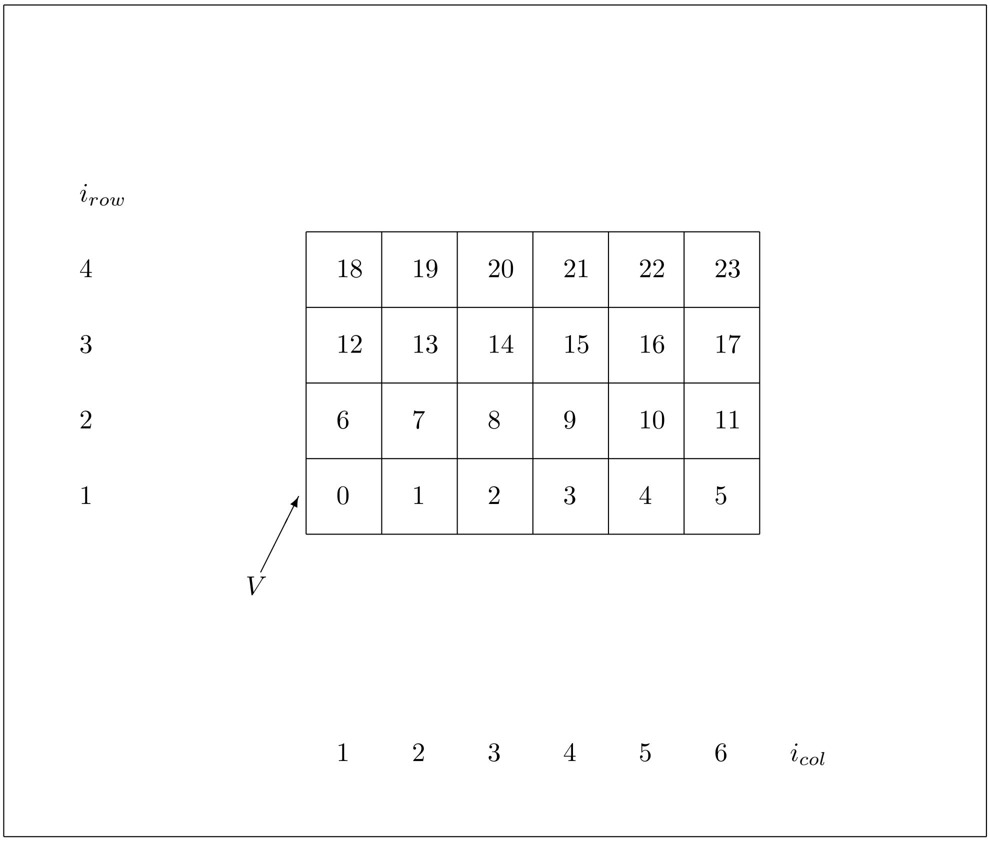

Figure 9: ANSI C - mesh ZR type D

- $V \equiv$ `double* ptr_V`
- `unsigned int size_row == 4`
- `unsigned int size_col == 6`
- `unsigned int i_row == 1, 2, .., 4`
- `unsigned int i_col == 1,2, .., 6`
- `double h == 1.0 [mm]`

13 Partial derivatives on Oz axis

13.1 Personal note

This is my presonal interpretation. I cannot guarantee correctness of this approach

13.2 Nodes numbering (on axis O_z)

We will try to work with P_2 point (determine approximations of aprtial derivatives for point P_2, which lies on axis O_z). Nodes numbering on axis O_z differs from numbering convention in Pierre Grivet's book.

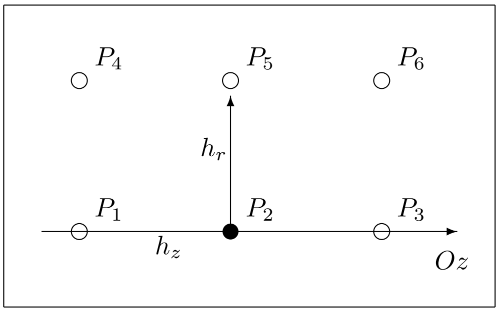

Figure 10: Nodes on axis Oz

Point P_2 is situated on O_z axis. It has 2 neighbours on axis O_z : P_1 and P_3. Node P_5 lies abowe P_2 node. The mesh step in r direction is h_r. The mesh step in z direction is h_z.

13.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates

Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates [3]:

$$\begin{aligned} V_{(z,r)} = V_{(z_0,r_0)} + \left(\frac{\partial V}{\partial z}\right)_{(z_0,r_0)} (z - z_0) + \\ \left(\frac{\partial V}{\partial r}\right)_{(z_0,r_0)} (r - r_0) + \\ \frac{1}{2!}\left(\frac{\partial^2 V}{\partial z^2}\right)_{(z_0,r_0)} (z - z_0)^2 + ... \end{aligned} \tag{13.1}$$

13.4 Laplace operator in rotationally symmetrical systems

Laplace operator in cylindrical coordinates (cylindrical symmetry) consist of 3 elements [1] (on page 42):

$$\begin{aligned}\nabla^2\left(V_{(z,r)}\right) = &\left(\frac{\partial^2 V}{\partial r^2}\right) + \\ &\frac{1}{r}\left(\frac{\partial V}{\partial r}\right) + \\ &\left(\frac{\partial^2 V}{\partial z^2}\right)\end{aligned} \tag{13.2}$$

In this chapter we will try to determine approximation of each term.

13.5 Value of first partial derivative of V with respect to r on axis Oz

In cylindrically symmetrical field first partial derivative of V (by r) on axis Oz equals zero (because $V_{(+\mathrm{d}r)} = V_{(-\mathrm{d}r)}$)

$$\left(\frac{\partial V}{\partial r}\right)_{(z,r=0)} = 0 \tag{13.3}$$

13.6 Value of second partial derivative of V with respect to r on axis Oz

In this subchapter we will try to determine the first term of equation13.2

In our case there is node P_2 on axis Oz. The nearest neihgbour of P_2 is node P_5, which lies „over Oz axis". The distance between P_2 and P_5 is h_r. When we „ walk away" axis Oz in r direction (from point P_2 to point P_5), the electric potential V_5 can be determined from truncated Taylor expansion 13.1 by expression:

$$V_5 \approx V_2 + \left(\frac{\partial V}{\partial r}\right)_{P_2} \cdot h_r + \frac{1}{2!}\left(\frac{\partial^2 V}{\partial r^2}\right)_{P_2} \cdot h_r^2 \tag{13.4}$$

We want to determine the second derivative:

$$\left(\frac{\partial^2 V}{\partial r^2}\right)_{P_2} =? \tag{13.5}$$

We solve equation 13.4 (using relation 13.3).

$$\left(\frac{\partial^2 V}{\partial r^2}\right)_{P_2} \approx \frac{2!\left(V_5 - V_2\right)}{h_r^2} = \frac{2\left(V_5 - V_2\right)}{h_r^2} \tag{13.6}$$

This is final form of approximation of the second derivative of V with respect to r on axis Oz. It will help us to determine Laplace operator in rotationally symmetrical systems.

13.7 Value of first partial derivative of V with respect to r divided by r on axis Oz

We will try to determine the second term of relation 13.2 Wheh we are on Oz axis, the second term has to be determined (because it aims to value $\frac{[0]}{[0]}$).

When we „ walk away" axis Oz in r direction, the electric potential $V_{(z_0,r)}$ can be determined from truncated Taylor expansion by:

$$V_{(z_0,r)} \approx V_{(z_0,r_0)} + \left(\frac{\partial V}{\partial r}\right)_{(z_0,r_0)} (r - r_0) \tag{13.7}$$

On Oz axis $r_0 = 0$, so $(r_0 - r) = r$

Thus we have:

$$V_{(z_0,r)} \approx V_{(z_0,0)} + \left(\frac{\partial V}{\partial r}\right)_{(z_0,0)} \cdot r \tag{13.8}$$

Now let us differentiate (both sides) of such relation:

$$| \, \frac{\partial}{\partial r} \tag{13.9}$$

We get:

$$\left(\frac{\partial V}{\partial r}\right)_{(z_0,r)} \approx \left(\frac{\partial V}{\partial r}\right)_{(z_0,0)} + \left(\frac{\partial^2 V}{\partial r^2}\right)_{(z_0,0)} \cdot r + \left(\frac{\partial V}{\partial r}\right)_{(z_0,0)} \cdot 1 \tag{13.10}$$

On axis Oz we can apply relation 13.3. That's why we can remove these two terms (first and third) from equation 13.10:

So we get (if $r = 0$):

$$\left(\frac{\partial V}{\partial r}\right)_{(z_0,r)} \approx \left(\frac{\partial^2 V}{\partial r^2}\right)_{(z_0,0)} \cdot r \tag{13.11}$$

We can now divide both sides by r.

$$| \cdot \frac{1}{r} \tag{13.12}$$

We have relation, which has been published in Pierre Grivet's book[1].

$$\left(\frac{1}{r}\frac{\partial V}{\partial r}\right)_{(z_0,r)} \approx \left(\frac{\partial^2 V}{\partial r^2}\right)_{(z_0,0)} \tag{13.13}$$

Approximation of this term on numerical mesh has been already determined in previous subsection (13.6).

13.8 Value of second partial derivative of V with respect to z on axis Oz

The third term of Laplace operator in rotationally symmetrical systems 13.2 takes form (on picture 10):

$$\left(\frac{\partial^2 V}{\partial z^2}\right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_z} - \frac{V_2 - V_1}{h_z}}{h_z} = \frac{V_1 + V_3 - 2V_2}{h_z^2} \tag{13.14}$$

Now we have determined all the 3 approximations o partial derivatives of V in cylindrically symetrical systems (on axis O_z).

14 Partial derivatives off Oz axis

14.1 Personal note

This is my presonal interpretation. I cannot guarantee correctness of this approach

14.2 Nodes numbering in Liebmann mesh (off axis O_z)

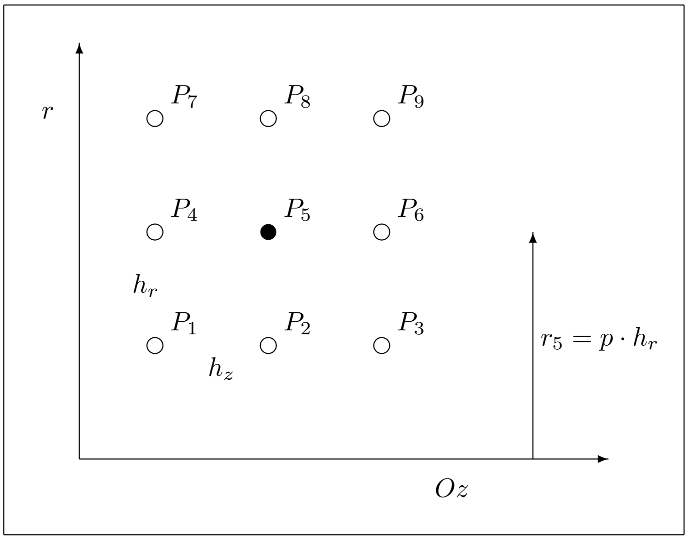

Figure 11: Nodes off axis Oz. Exemplary vector r_5 describes distance from axis O_z to node P_5

Mesh step in z direction is h_z. Mesh step in r direction is h_r. Sample mesh points P_5 lies off O_z axis. Distance between mesh point P_5 and O_z axis is r_5.

For ANSI C meshes (in Liebmann source code) the following relations have place:

$$r = ph_r \tag{14.1}$$

$$p = i_{row} - 1 \tag{14.2}$$

14.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates

Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates [3]:

$$
\begin{aligned}
V_{(z,r)} = V_{(z_0,r_0)} + \left(\frac{\partial V}{\partial z}\right)_{(z_0,r_0)} (z - z_0) + \\
\left(\frac{\partial V}{\partial r}\right)_{(z_0,r_0)} (r - r_0) + \\
\frac{1}{2!}\left(\frac{\partial^2 V}{\partial z^2}\right)_{(z_0,r_0)} (z - z_0)^2 + ...
\end{aligned}
\tag{14.3}
$$

14.4 Laplace operator in rotationally symmetrical systems

Laplace operator in cylindrical coordinates (cylindrical symmetry) consist of 3 elements [1] (on page 42):

$$
\begin{aligned}
\nabla^2 \left(V_{(z,r)}\right) = \left(\frac{\partial^2 V}{\partial r^2}\right) + \\
\frac{1}{r}\left(\frac{\partial V}{\partial r}\right) + \\
\left(\frac{\partial^2 V}{\partial z^2}\right)
\end{aligned}
\tag{14.4}
$$

In this chapter we will try to determine approximation of each term.

14.5 Value of second partial derivative of V with respect to r off axis Oz

$$
\left(\frac{\partial^2 V}{\partial r^2}\right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_r} - \frac{V_5 - V_2}{h_r}}{h_r} = \frac{V_2 + V_8 - 2V_5}{h_r^2} \tag{14.5}
$$

14.6 Value of first partial derivative of V with respect to r divided by r off axis Oz

$$
\frac{1}{r_5}\left(\frac{\partial V}{\partial r}\right)_{P_5} \approx \frac{1}{r_5}\frac{V_8 - V_2}{2h_r} = \frac{V_8 - V_2}{2r_5 h_z} \tag{14.6}
$$

14.7 Value of second partial derivative of V with respect to z off axis Oz

$$
\left(\frac{\partial^2 V}{\partial z^2}\right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_z} - \frac{V_5 - V_4}{h_z}}{h_z} = \frac{V_4 + V_6 - 2V_5}{h_z^2} \tag{14.7}
$$

15 Relaxation formula for node P1 (on axis Oz)

15.1 Node description

Left, bottom corner of mesh ZR (on axis Oz).

15.2 Calculation of relaxation formula

Laplace equation at node P_1

$$\nabla^2 \left(V_{(z,r)}\right)_{P_1} = 0 \tag{15.1}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_1} + \left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_1} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_1} = 0 \tag{15.2}$$

Approximation of partial derivatives of $V_{(z,r)}$ at node P_1

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_1} \approx \frac{\frac{V_4 - V_1}{h_r} - \frac{V_1 - V_4}{h_r}}{h_r} = \frac{2\left(V_4 - V_1\right)}{h_r^2} \tag{15.3}$$

$$\left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_1} \approx \frac{2\left(V_4 - V_1\right)}{h_r^2} \tag{15.4}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_1} \approx \frac{\frac{V_2 - V_1}{h_z} - \frac{V_1 - V_{1z-}}{h_z}}{h_z} = \frac{V_2 - V_1}{h_z^2} - \frac{g_{1z-}}{h_z} \tag{15.5}$$

Let us substitute approximations to Laplace equation.

$$\frac{2(V_4 - V_1)}{h_r^2} + \frac{2(V_4 - V_1)}{h_r^2} + \frac{V_2 - V_1}{h_z^2} - \frac{g_{1z-}}{h_z} = 0 \tag{15.6}$$

$$\frac{2(V_4 - V_1)}{h_r^2} + \frac{2(V_4 - V_1)}{h_r^2} + \frac{V_2 - V_1}{h_z^2} = \frac{g_{1z-}}{h_z} \tag{15.7}$$

Let us find V_1

$$V_1 = ? \tag{15.8}$$

Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \tag{15.9}$$

We obtain

$$\begin{split}2V_4 h_z^2 - 2V_1 h_z^2 + 2V_4 h_z^2 - 2V_1 h_z^2 + V_2 h_r^2 - V_1 h_r^2 = \\ g_{1z-} h_z h_r^2\end{split} \tag{15.10}$$

Let us simplify this equation:

$$V_1\left(2h_z^2 + 2h_z^2 + h_r^2\right) = \\ 2V_4h_z^2 + 2V_4h_z^2 + V_2h_r^2 - g_{1z-}h_zh_r^2 \tag{15.11}$$

So we have:

$$V_1\left(4h_z^2 + h_r^2\right) = 4V_4h_z^2 + V_2h_r^2 - g_{1z-}h_zh_r^2 \tag{15.12}$$

15.3 Final forms of relaxation formula

15.3.1 zrLV_RELAX5_P1_ON_A

$$h_z \neq h_r$$
$$g_{1z-} \neq 0$$
$$V_1 = \frac{4V_4h_z^2 + V_2h_r^2 - g_{1z-}h_zh_r^2}{4h_z^2 + h_r^2} \tag{15.13}$$

15.3.2 zrLV_RELAX5_P1_ON_B

$$h_z \neq h_r$$
$$g_{1z-} = 0$$
$$V_1 = \frac{4V_4h_z^2 + V_2h_r^2}{4h_z^2 + h_r^2} \tag{15.14}$$

15.3.3 zrLV_RELAX5_P1_ON_C

$$h_z = h_r = h$$
$$g_{1z-} \neq 0$$
$$V_1 = \frac{4V_4 + V_2 - g_{1z-}h}{5} \tag{15.15}$$

15.3.4 zrLV_RELAX5_P1_ON_D

$$h_z = h_r = h$$
$$g_{1z-} = 0$$
$$V_1 = \frac{4V_4 + V_2}{5} \tag{15.16}$$

16 Relaxation formula for node P2 (on axis Oz)

16.1 Node description

Bottom edge of mesh ZR (on axis Oz).

16.2 Calculation of relaxation formula

Laplace equation at node P_2

$$\nabla^2 \left(V_{(z,r)}\right)_{P_2} = 0 \tag{16.1}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_2} + \left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_2} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_2} = 0 \tag{16.2}$$

Approximation of partial derivatives of $V_{(z,r)}$ at node P_2

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_2} \approx \frac{\frac{V_5 - V_2}{h_r} - \frac{V_2 - V_5}{h_r}}{h_r} = \frac{2\left(V_5 - V_2\right)}{h_r^2} \tag{16.3}$$

$$\left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_2} \approx \frac{2\left(V_5 - V_2\right)}{h_r^2} \tag{16.4}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_z} - \frac{V_2 - V_1}{h_z}}{h_z} = \frac{V_1 + V_3 - 2V_2}{h_z^2} \tag{16.5}$$

Let us substitute approximations to Laplace equation.

$$\frac{2(V_5 - V_2)}{h_r^2} + \frac{2(V_5 - V_2)}{h_r^2} + \frac{V_1 + V_3 - 2V_2}{h_z^2} = 0 \tag{16.6}$$

There are no g expressions to move, to formula 7 has identical form as formula 6.

$$\frac{2(V_5 - V_2)}{h_r^2} + \frac{2(V_5 - V_2)}{h_r^2} + \frac{V_1 + V_3 - 2V_2}{h_z^2} = 0 \tag{16.7}$$

Let us find V_2

$$V_2 = ? \tag{16.8}$$

Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \tag{16.9}$$

We obtain

$$2V_5 h_z^2 - 2V_2 h_z^2 + 2V_5 h_z^2 - 2V_2 h_z^2 + V_1 h_r^2 + V_3 h_r^2 - 2V_2 h_r^2 = 0 \tag{16.10}$$

Let us simplify this equation:

$$V_2 \left(2h_z^2 + 2h_z^2 + 2h_r^2\right) = 2V_5h_z^2 + 2V_5h_z^2 + V_1h_r^2 + V_3h_r^2 \tag{16.11}$$

So we have:

$$V_2 \left(4h_z^2 + 2h_r^2\right) = 4V_5h_z^2 + (V_1 + V_3)h_r^2 \tag{16.12}$$

16.3 Final forms of relaxation formula

16.3.1 zrLV_RELAX5_P2_ON_A

$$h_z \neq h_r$$

$$V_2 = \frac{4V_5h_z^2 + (V_1 + V_3)h_r^2}{4h_z^2 + 2h_r^2} \tag{16.13}$$

16.3.2 zrLV_RELAX5_P2_ON_B

$$h_z \neq h_r$$

$$V_2 = \frac{4V_5h_z^2 + (V_1 + V_3)h_r^2}{4h_z^2 + 2h_r^2} \tag{16.14}$$

16.3.3 zrLV_RELAX5_P2_ON_C

$$h_z = h_r = h$$

$$V_2 = \frac{4V_5 + V_1 + V_3}{6} \tag{16.15}$$

16.3.4 zrLV_RELAX5_P2_ON_D

$$h_z = h_r = h$$

$$V_2 = \frac{4V_5 + V_1 + V_3}{6} \tag{16.16}$$

17 Relaxation formula for node P3 (on axis Oz)

17.1 Node description

Right, bottom corner of mesh ZR (on axis Oz).

17.2 Calculation of relaxation formula

Laplace equation at node P_3

$$\nabla^2 \left(V_{(z,r)}\right)_{P_3} = 0 \tag{17.1}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_3} + \left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_3} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_3} = 0 \tag{17.2}$$

Approximation of partial derivatives of $V_{(z,r)}$ at node P_3

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_3} \approx \frac{\frac{V_6 - V_3}{h_r} - \frac{V_3 - V_6}{h_r}}{h_r} = \frac{2\left(V_6 - V_3\right)}{h_r^2} \tag{17.3}$$

$$\left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_3} \approx \frac{2\left(V_6 - V_3\right)}{h_r^2} \tag{17.4}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_3} \approx \frac{\frac{V_{3z+} - V_3}{h_z} - \frac{V_3 - V_2}{h_z}}{h_z} = \frac{V_2 - V_3}{h_z^2} + \frac{g_{3z+}}{h_z} \tag{17.5}$$

Let us substitute approximations to Laplace equation.

$$\frac{2(V_6 - V_3)}{h_r^2} + \frac{2(V_6 - V_3)}{h_r^2} + \frac{V_2 - V_3}{h_z^2} + \frac{g_{3z+}}{h_z} = 0 \tag{17.6}$$

$$\frac{2(V_6 - V_3)}{h_r^2} + \frac{2(V_6 - V_3)}{h_r^2} + \frac{V_2 - V_3}{h_z^2} = -\frac{g_{3z+}}{h_z} \tag{17.7}$$

Let us find V_3

$$V_3 = ? \tag{17.8}$$

Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \tag{17.9}$$

We obtain

$$\begin{split} 2V_6 h_z^2 - 2V_3 h_z^2 + 2V_6 h_z^2 - 2V_3 h_z^2 + V_2 h_r^2 - V_3 h_r^2 = \\ -g_{3z+} h_z h_r^2 \end{split} \tag{17.10}$$

Let us simplify this equation:

$$\begin{aligned} V_3\left(2h_z^2 + 2h_z^2 + h_r^2\right) = \\ 2V_6h_z^2 + 2V_6h_z^2 + V_2h_r^2 + g_{3z+}h_zh_r^2 \end{aligned} \tag{17.11}$$

So we have:

$$V_3\left(4h_z^2 + h_r^2\right) = 4V_6h_z^2 + V_2h_r^2 + g_{1z-}h_zh_r^2 \tag{17.12}$$

17.3 Final forms of relaxation formula

17.3.1 zrLV_RELAX5_P3_ON_A

$$h_z \neq h_r$$
$$g_{3z+} \neq 0$$
$$V_3 = \frac{4V_6h_z^2 + V_2h_r^2 + g_{3z+}h_zh_r^2}{4h_z^2 + h_r^2} \tag{17.13}$$

17.3.2 zrLV_RELAX5_P3_ON_B

$$h_z \neq h_r$$
$$g_{3z+} = 0$$
$$V_3 = \frac{4V_6h_z^2 + V_2h_r^2}{4h_z^2 + h_r^2} \tag{17.14}$$

17.3.3 zrLV_RELAX5_P3_ON_C

$$h_z = h_r = h$$
$$g_{3z+} \neq 0$$
$$V_3 = \frac{4V_6 + V_2 + g_{3z+}h}{5} \tag{17.15}$$

17.3.4 zrLV_RELAX5_P3_ON_D

$$h_z = h_r = h$$
$$g_{3z+} = 0$$
$$V_3 = \frac{4V_6 + V_2}{5} \tag{17.16}$$

18 Relaxation formula for node P4

18.1 Node description

Left edge of mesh ZR.

18.2 Calculation of relaxation formula

Laplace equation at node P_4

$$\nabla^2 \left(V_{(z,r)}\right)_{P_4} = 0 \tag{18.1}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_4} + \left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_4} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_4} = 0 \tag{18.2}$$

Approximation of partial derivatives of $V_{(z,r)}$ at node P_4

(note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_4} \approx \frac{\frac{V_7 - V_4}{h_r} - \frac{V_4 - V_1}{h_r}}{h_r} = \frac{V_1 + V_7 - 2V_4}{h_r^2} \tag{18.3}$$

$$\left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_4} \approx \frac{1}{r}\frac{V_7 - V_1}{2h_r} = \frac{V_7 - V_1}{2ph_r^2} \tag{18.4}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_4} \approx \frac{\frac{V_5 - V_1}{h_z} - \frac{V_4 - V_{4z-}}{h_z}}{h_z} = \frac{V_5 - V_4}{h_z^2} - \frac{g_{4z-}}{h_z} \tag{18.5}$$

Let us substitute approximations to Laplace equation.

$$\frac{V_1 + V_7 - 2V_4}{h_r^2} + \frac{V_7 - V_1}{2ph_r^2} + \frac{V_5 - V_4}{h_z^2} - \frac{g_{4z-}}{h_z} = 0 \tag{18.6}$$

$$\frac{V_1 + V_7 - 2V_4}{h_r^2} + \frac{V_7 - V_1}{2ph_r^2} + \frac{V_5 - V_4}{h_z^2} = \frac{g_{4z-}}{h_z} \tag{18.7}$$

Let us find V_4

$$V_4 = ? \tag{18.8}$$

Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \tag{18.9}$$

We obtain

$$2pV_1h_z^2 + 2pV_7h_z^2 - 4pV_4h_z^2 + V_7h_z^2 - V_1h_z^2 + \\ + 2pV_5h_r^2 - 2pV_4h_r^2 = 2pg_{4z-}h_zh_r^2 \tag{18.10}$$

Let us simplify this equation:

$$V_4\left(4ph_z^2 + 2ph_r^2\right) = V_1(2ph_z^2 - h_z^2) + V_7(2ph_z^2 + h_z^2) + V_5 2ph_r^2 - \\ 2pg_{4z-}h_zh_r^2 \tag{18.11}$$

So we have:

$$V_4 2p(2h_z^2 + h_r^2) = V_1h_z^2(2p-1) + V_7h_z^2(2p+1) + V_5 2ph_r^2 - 2pg_{4z-}h_zh_r^2 \tag{18.12}$$

18.3 Final forms of relaxation formula

18.3.1 zrLV_RELAX5_P4_A

$$h_z \neq h_r$$
$$g_{4z-} \neq 0$$
$$V_4 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_1 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_7 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 - \frac{g_{4z-}h_zh_r^2}{2h_z^2 + h_r^2} \tag{18.13}$$

18.3.2 zrLV_RELAX5_P4_B

$$h_z \neq h_r$$
$$g_{4z-} = 0$$
$$V_4 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_1 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_7 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 \tag{18.14}$$

18.3.3 zrLV_RELAX5_P4_C

$$h_z = h_r = h$$
$$g_{4z-} \neq 0$$
$$V_4 = \frac{2p-1}{6p}V_1 + \frac{2p+1}{6p}V_7 + \frac{1}{3}V_5 - \frac{g_{4z-}h}{3} \tag{18.15}$$

18.3.4 zrLV_RELAX5_P4_D

$$h_z = h_r = h$$
$$g_{4z-} = 0$$
$$V_4 = \frac{2p-1}{6p}V_1 + \frac{2p+1}{6p}V_7 + \frac{1}{3}V_5 \tag{18.16}$$

19 Relaxation formula for node P5

19.1 Node description

Inner node of mesh ZR.

19.2 Calculation of relaxation formula

Laplace equation at node P_5

$$\nabla^2 \left(V_{(z,r)}\right)_{P_5} = 0 \tag{19.1}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_5} + \left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_5} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_5} = 0 \tag{19.2}$$

Approximation of partial derivatives of $V_{(z,r)}$ at node P_5

(note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_r} - \frac{V_5 - V_2}{h_r}}{h_r} = \frac{V_2 + V_8 - 2V_5}{h_r^2} \tag{19.3}$$

$$\left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_5} \approx \frac{1}{r}\frac{V_8 - V_2}{2h_r} = \frac{V_8 - V_2}{2ph_r^2} \tag{19.4}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_z} - \frac{V_5 - V_4}{h_z}}{h_z} = \frac{V_4 + V_6 - 2V_5}{h_z^2} \tag{19.5}$$

Let us substitute approximations to Laplace equation.

$$\frac{V_2 + V_8 - 2V_5}{h_r^2} + \frac{V_8 - V_2}{2ph_r^2} + \frac{V_4 + V_6 - 2V_5}{h_z^2} = 0 \tag{19.6}$$

We don't need to simplify this equation in step 7:

$$\frac{V_2 + V_8 - 2V_5}{h_r^2} + \frac{V_8 - V_2}{2ph_r^2} + \frac{V_4 + V_6 - 2V_5}{h_z^2} = 0 \tag{19.7}$$

Let us find V_5

$$V_5 = ? \tag{19.8}$$

Let us multiply both sides

$$\mid \cdot 2ph_z^2 h_r^2 \tag{19.9}$$

We obtain

$$
\begin{aligned}
2pV_2h_z^2 + 2pV_8h_z^2 - 4pV_5h_z^2 + V_8h_z^2 - V_2h_z^2 + \\
+2pV_4h_r^2 + 2pV_6h_r^2 - 4pV_5h_r^2 = 0
\end{aligned}
\tag{19.10}
$$

Let us simplify this equation:

$$
\begin{aligned}
V_5\left(4ph_z^2 + 4ph_r^2\right) = V_2(2ph_z^2 - h_z^2) + V_8(2ph_z^2 + h_z^2) + \\
+2ph_r^2V_4 + 2ph_r^2V_6
\end{aligned}
\tag{19.11}
$$

So we have:

$$
V_5 4p(h_z^2 + h_r^2) = V_2h_z^2(2p-1) + V_8h_z^2(2p+1) + V_4 2ph_r^2 + V_6 2ph_r^2 \tag{19.12}
$$

19.3 Final forms of relaxation formula

19.3.1 zrLV_RELAX5_P5_A

$$
h_z \neq h_r
$$

$$
V_5 = \frac{h_z^2(2p-1)}{4p(h_z^2+h_r^2)}V_2 + \frac{h_z^2(2p+1)}{4p(h_z^2+h_r^2)}V_8 + \frac{h_r^2}{2(h_z^2+h_r^2)}(V_4+V_6) \tag{19.13}
$$

19.3.2 zrLV_RELAX5_P5_B

This formula is identica to formula A:

$$
h_z \neq h_r
$$

$$
V_5 = \frac{h_z^2(2p-1)}{4p(h_z^2+h_r^2)}V_2 + \frac{h_z^2(2p+1)}{4p(h_z^2+h_r^2)}V_8 + \frac{h_r^2}{2(h_z^2+h_r^2)}(V_4+V_6) \tag{19.14}
$$

19.3.3 zrLV_RELAX5_P5_C

$$
h_z = h_r = h
$$

$$
V_5 = \frac{2p-1}{8p}V_2 + \frac{2p+1}{8p}V_8 + \frac{1}{4}(V_4+V_6) \tag{19.15}
$$

19.3.4 zrLV_RELAX5_P5_D

This formula is identical to formula C:

$$
h_z = h_r = h
$$

$$
V_5 = \frac{2p-1}{8p}V_2 + \frac{2p+1}{8p}V_8 + \frac{1}{4}(V_4+V_6) \tag{19.16}
$$

20 Relaxation formula for node P6

20.1 Node description

Right edge of mesh ZR.

20.2 Calculation of relaxation formula

Laplace equation at node P_6

$$\nabla^2 \left(V_{(z,r)}\right)_{P_6} = 0 \tag{20.1}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_6} + \left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_6} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_6} = 0 \tag{20.2}$$

Approximation of partial derivatives of $V_{(z,r)}$ at node P_6

(note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_6} \approx \frac{\frac{V_9 - V_6}{h_r} - \frac{V_6 - V_3}{h_r}}{h_r} = \frac{V_3 + V_9 - 2V_6}{h_r^2} \tag{20.3}$$

$$\left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_6} \approx \frac{1}{r}\frac{V_9 - V_3}{2h_r} = \frac{V_9 - V_3}{2ph_r^2} \tag{20.4}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_6} \approx \frac{\frac{V_{6z+} - V_6}{h_z} - \frac{V_6 - V_5}{h_z}}{h_z} = \frac{V_5 - V_6}{h_z^2} + \frac{g_{6z+}}{h_z} \tag{20.5}$$

Let us substitute approximations to Laplace equation.

$$\frac{V_3 + V_9 - 2V_6}{h_r^2} + \frac{V_9 - V_3}{2ph_r^2} + \frac{V_5 - V_6}{h_z^2} + \frac{g_{6z+}}{h_z} = 0 \tag{20.6}$$

$$\frac{V_3 + V_9 - 2V_6}{h_r^2} + \frac{V_9 - V_3}{2ph_r^2} + \frac{V_5 - V_6}{h_z^2} = -\frac{g_{6z+}}{h_z} \tag{20.7}$$

Let us find V_6

$$V_6 = ? \tag{20.8}$$

Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \tag{20.9}$$

We obtain

$$
\begin{aligned}
2pV_3h_z^2 + 2pV_9h_z^2 - 4pV_6h_z^2 + V_9h_z^2 - V_3h_z^2 + \\
+2pV_5h_r^2 - 2pV_6h_r^2 = -2pg_{6z+}h_zh_r^2
\end{aligned}
\tag{20.10}
$$

Let us simplify this equation:

$$
\begin{aligned}
V_6\left(4ph_z^2 + 2ph_r^2\right) = V_3(2ph_z^2 - h_z^2) + V_9(2ph_z^2 + h_z^2) + V_5 2ph_r^2 + \\
2pg_{6z+}h_zh_r^2
\end{aligned}
\tag{20.11}
$$

So we have:

$$
V_6 2p(2h_z^2 + h_r^2) = V_3h_z^2(2p-1) + V_9h_z^2(2p+1) + V_5 2ph_r^2 + 2pg_{6z+}h_zh_r^2 \tag{20.12}
$$

20.3 Final forms of relaxation formula

20.3.1 zrLV_RELAX5_P6_A

$$
h_z \neq h_r
$$
$$
g_{6z+} \neq 0
$$
$$
V_6 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_3 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_9 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 + \frac{g_{6z+}h_zh_r^2}{2h_z^2 + h_r^2} \tag{20.13}
$$

20.3.2 zrLV_RELAX5_P6_B

$$
h_z \neq h_r
$$
$$
g_{6z+-} = 0
$$
$$
V_6 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_3 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_9 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 \tag{20.14}
$$

20.3.3 zrLV_RELAX5_P6_C

$$
h_z = h_r = h
$$
$$
g_{6z+} \neq 0
$$
$$
V_6 = \frac{2p-1}{6p}V_3 + \frac{2p+1}{6p}V_9 + \frac{1}{3}V_5 + \frac{g_{6z+}h}{3} \tag{20.15}
$$

20.3.4 zrLV_RELAX5_P6_D

$$
h_z = h_r = h
$$
$$
g_{6z+} = 0
$$
$$
V_4 = \frac{2p-1}{6p}V_3 + \frac{2p+1}{6p}V_9 + \frac{1}{3}V_5 \tag{20.16}
$$

21 Relaxation formula for node P7

21.1 Node description

Left, upper corner of mesh ZR.

21.2 Calculation of relaxation formula

Laplace equation at node P_7

$$\nabla^2 \left(V_{(z,r)}\right)_{P_7} = 0 \tag{21.1}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_7} + \left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_7} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_7} = 0 \tag{21.2}$$

Approximation of partial derivatives of $V_{(z,r)}$ at node P_7

(note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_7} \approx \frac{\frac{V_{7r+}-V_7}{h_r} - \frac{V_7-V_4}{h_r}}{h_r} = \frac{V_4 - V_7}{h_r^2} + \frac{g_{7r+}}{h_r} \tag{21.3}$$

$$\left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_7} \approx \frac{1}{r}\frac{V_{7r+} - V_4}{2h_r} = \frac{V_7 + g_{7r+}h_r - V_4}{2ph_r^2} = \frac{V_7 - V_4}{2ph_r^2} + \frac{g_{7r+}}{2ph_r} \tag{21.4}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_7} \approx \frac{\frac{V_8-V_7}{h_z} - \frac{V_7-V_{7z-}}{h_z}}{h_z} = \frac{V_8 - V_7}{h_z^2} - \frac{g_{7z-}}{h_z} \tag{21.5}$$

Let us substitute approximations to Laplace equation.

$$\frac{V_4 - V_7}{h_r^2} + \frac{g_{7r+}}{h_r} + \frac{V_7 - V_4}{2ph_r^2} + \frac{g_{7r+}}{2ph_r} + \frac{V_8 - V_7}{h_z^2} - \frac{g_{7z-}}{h_z} = 0 \tag{21.6}$$

$$\frac{V_4 - V_7}{h_r^2} + \frac{V_7 - V_4}{2ph_r^2} + \frac{V_8 - V_7}{h_z^2} = -\frac{g_{7r+}}{h_r} - \frac{g_{7r+}}{2ph_r} + \frac{g_{7z-}}{h_z} \tag{21.7}$$

Let us find V_7

$$V_7 =? \tag{21.8}$$

Let us multiply both sides

$$| \cdot 2ph_z^2h_r^2 \tag{21.9}$$

We obtain

$$\begin{aligned} 2pV_4h_z^2 - 2pV_7h_z^2 + V_7h_z^2 - V_4h_z^2 + 2pV_8h_r^2 - 2pV_7h_r^2 = \\ -2pg_{7r+}h_z^2h_r - g_{7r+}h_z^2h_r + 2pg_{7z-}h_zh_r^2 \end{aligned} \tag{21.10}$$

Let us simplify this equation:

$$\begin{aligned} V_7\left(2ph_z^2 - h_z^2 + 2ph_r^2\right) = V_4(2ph_z^2 - h_z^2) + V_8(2ph_r^2) + \\ 2pg_{7r+}h_z^2h_r + g_{7r+}h_z^2h_r - 2pg_{7z-}h_zh_r^2 \end{aligned} \tag{21.11}$$

So we have:

$$\begin{aligned} V_7((2p-1)h_z^2 + 2ph_r^2) = V_4h_z^2(2p-1) + V_8 2ph_r^2 + \\ 2pg_{7r+}h_z^2h_r + g_{7r+}h_z^2h_r - 2pg_{7z-}h_zh_r^2 \end{aligned} \tag{21.12}$$

21.3 Final forms of relaxation formula

21.3.1 zrLV_RELAX5_P7_A

$$h_z \neq h_r$$
$$g_{7z-} \neq 0$$
$$g_{7r+} \neq 0$$

$$\begin{aligned} V_7 = \frac{h_z^2(2p-1)}{(2p-1)h_z^2 + 2ph_r^2}V_4 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2}V_8 + \\ \frac{(2p+1)g_{7r+}h_z^2h_r - 2pg_{7z-}h_zh_r^2}{(2p-1)h_z^2 + 2ph_r^2} \end{aligned} \tag{21.13}$$

21.3.2 zrLV_RELAX5_P7_B

$$h_z \neq h_r$$
$$g_{7z-} = 0$$
$$g_{7r+} = 0$$

$$V_7 = \frac{h_z^2(2p-1)}{(2p-1)h_z^2 + 2ph_r^2}V_4 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2}V_8 \tag{21.14}$$

21.3.3 zrLV_RELAX5_P7_C

$$h_z = h_r = h$$
$$g_{7z-} \neq 0$$
$$g_{7r+} \neq 0$$
$$V_7 = \frac{2p-1}{4p-1}V_4 + \frac{2p}{4p-1}V_8 + \frac{h((2p+1)g_{7r+} - g_{7z-})}{4p-1} \tag{21.15}$$

21.3.4 zrLV_RELAX5_P7_D

$$h_z = h_r = h$$
$$g_{7z-} = 0$$
$$g_{7r+} = 0$$
$$V_7 = \frac{2p-1}{4p-1}V_4 + \frac{2p}{4p-1}V_8 \tag{21.16}$$

22 Relaxation formula for node P8

22.1 Node description

Upper edge of mesh ZR.

22.2 Calculation of relaxation formula

Laplace equation at node P_8

$$\nabla^2 \left(V_{(z,r)}\right)_{P_8} = 0 \tag{22.1}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_8} + \left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_8} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_8} = 0 \tag{22.2}$$

Approximation of partial derivatives of $V_{(z,r)}$ at node P_8

(note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_8} \approx \frac{\frac{V_{8r+} - V_8}{h_r} - \frac{V_8 - V_5}{h_r}}{h_r} = \frac{V_5 - V_8}{h_r^2} + \frac{g_{8r+}}{h_r} \tag{22.3}$$

$$\left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_8} \approx \frac{1}{r}\frac{V_{8r+} - V_5}{2h_r} = \frac{V_8 + g_{8r+}h_r - V_5}{2ph_r^2} = \frac{V_8 - V_5}{2ph_r^2} + \frac{g_{8r+}}{2ph_r} \tag{22.4}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_8} \approx \frac{\frac{V_9 - V_8}{h_z} - \frac{V_8 - V_7}{h_z}}{h_z} = \frac{V_7 + V_9 - 2V_8}{h_z^2} \tag{22.5}$$

Let us substitute approximations to Laplace equation.

$$\frac{V_5 - V_8}{h_r^2} + \frac{g_{8r+}}{h_r} + \frac{V_8 - V_5}{2ph_r^2} + \frac{g_{8r+}}{2ph_r} + \frac{V_7 + V_9 - 2V_8}{h_z^2} = 0 \tag{22.6}$$

$$\frac{V_5 - V_8}{h_r^2} + \frac{V_8 - V_5}{2ph_r^2} + \frac{V_7 + V_9 - 2V_8}{h_z^2} = -\frac{g_{8r+}}{h_r} - \frac{g_{8r+}}{2ph_r} \tag{22.7}$$

Let us find V_8

$$V_8 = ? \tag{22.8}$$

Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \tag{22.9}$$

We obtain

$$\begin{split} 2pV_5h_z^2 - 2pV_8h_z^2 + V_8h_z^2 - V_5h_z^2 + 2pV_7h_r^2 + 2pV_9h_r^2 - 4pV_8h_r^2 = \\ -2pg_{8r+}h_z^2h_r - g_{8r+}h_z^2h_r \end{split} \tag{22.10}$$

Let us simplify this equation:

$$\begin{split} V_8\left(2ph_z^2 - h_z^2 + 4ph_r^2\right) = V_5(2ph_z^2 - h_z^2) + (V_7 + V_9)2ph_r^2 + \\ 2pg_{8r+}h_z^2h_r + g_{8r+}h_z^2h_r \end{split} \tag{22.11}$$

So we have:

$$\begin{split} V_8((2p-1)h_z^2 + 4ph_r^2) = V_5h_z^2(2p-1) + (V_7 + V_9)2ph_r^2 + \\ 2pg_{8r+}h_z^2h_r + g_{8r+}h_z^2h_r \end{split} \tag{22.12}$$

22.3 Final forms of relaxation formula

22.3.1 zrLV_RELAX5_P8_A

$$h_z \neq h_r$$
$$g_{8r+} \neq 0$$
$$\begin{split} V_8 = \frac{h_z^2(2p-1)}{(2p-1)h_z^2 + 4ph_r^2}V_5 + \frac{2ph_r^2}{(2p-1)h_z^2 + 4ph_r^2}(V_7 + V_9) + \\ \frac{(2p+1)h_z^2h_rg_{8r+}}{(2p-1)h_z^2 + 4ph_r^2} \end{split} \tag{22.13}$$

22.3.2 zrLV_RELAX5_P8_B

$$h_z \neq h_r$$
$$g_{8r+} = 0$$
$$V_8 = \frac{h_z^2(2p-1)}{(2p-1)h_z^2 + 4ph_r^2}V_5 + \frac{2ph_r^2}{(2p-1)h_z^2 + 4ph_r^2}(V_7 + V_9) \tag{22.14}$$

22.3.3 zrLV_RELAX5_P8_C

$$h_z = h_r = h$$
$$g_{8r+} \neq 0$$
$$\begin{split} V_8 = \frac{2p-1}{6p-1}V_5 + \frac{2p}{6p-1}(V_7 + V_9) + \\ \frac{(2p+1)hg_{8r+}}{6p-1} \end{split} \tag{22.15}$$

22.3.4 zrLV_RELAX5_P8_D

$$h_z = h_r = h$$
$$g_{8r+} = 0$$
$$V_8 = \frac{2p-1}{6p-1} V_5 + \frac{2p}{6p-1}(V_7 + V_9) \tag{22.16}$$

23 Relaxation formula for node P9

23.1 Node description

Right, upper corner of mesh ZR.

23.2 Calculation of relaxation formula

Laplace equation at node P_9

$$\nabla^2 \left(V_{(z,r)}\right)_{P_9} = 0 \tag{23.1}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_9} + \left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_9} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_9} = 0 \tag{23.2}$$

Approximation of partial derivatives of $V_{(z,r)}$ at node P_9

(note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2}\right)_{P_9} \approx \frac{\frac{V_{9r+} - V_9}{h_r} - \frac{V_9 - V_6}{h_r}}{h_r} = \frac{V_6 - V_9}{h_r^2} + \frac{g_{9r+}}{h_r} \tag{23.3}$$

$$\left(\frac{1}{r}\frac{\partial V_{(z,r)}}{\partial r}\right)_{P_9} \approx \frac{1}{r}\frac{V_{9r+} - V_6}{2h_r} = \frac{V_9 + g_{9r+}h_r - V_6}{2ph_r^2} = \frac{V_9 - V_6}{2ph_r^2} + \frac{g_{9r+}}{2ph_r} \tag{23.4}$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2}\right)_{P_9} \approx \frac{\frac{V_{9z+} - V_9}{h_z} - \frac{V_9 - V_8}{h_z}}{h_z} = \frac{g_{9z-}}{h_z} + \frac{V_8 - V_9}{h_z^2} \tag{23.5}$$

Let us substitute approximations to Laplace equation.

$$\frac{V_6 - V_9}{h_r^2} + \frac{g_{9r+}}{h_r} + \frac{V_9 - V_6}{2ph_r^2} + \frac{g_{9r+}}{2ph_r} + \frac{V_8 - V_9}{h_z^2} + \frac{g_{9z+}}{h_z} = 0 \tag{23.6}$$

$$\frac{V_6 - V_9}{h_r^2} + \frac{V_9 - V_6}{2ph_r^2} + \frac{V_8 - V_9}{h_z^2} = -\frac{g_{9r+}}{h_r} - \frac{g_{9r+}}{2ph_r} - \frac{g_{9z+}}{h_z} \tag{23.7}$$

Let us find V_9

$$V_9 = ? \tag{23.8}$$

Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \tag{23.9}$$

We obtain

$$\begin{aligned} 2pV_6h_z^2 - 2pV_9h_z^2 + V_9h_z^2 - V_6h_z^2 + 2pV_8h_r^2 - 2pV_9h_r^2 = \\ -2pg_{9r+}h_z^2h_r - g_{9r+}h_z^2h_r - 2pg_{9z+}h_zh_r^2 \end{aligned} \tag{23.10}$$

Let us simplify this equation:

$$\begin{aligned} V_9\left(2ph_z^2 - h_z^2 + 2ph_r^2\right) = V_6(2ph_z^2 - h_z^2) + V_8(2ph_r^2) + \\ 2pg_{9r+}h_z^2h_r + g_{9r+}h_z^2h_r + 2pg_{9z+}h_zh_r^2 \end{aligned} \tag{23.11}$$

So we have:

$$\begin{aligned} V_9((2p-1)h_z^2 + 2ph_r^2) = V_6h_z^2(2p-1) + V_82ph_r^2 + \\ (2p+1)g_{9r+}h_z^2h_r + 2pg_{9z+}h_zh_r^2 \end{aligned} \tag{23.12}$$

23.3 Final forms of relaxation formula

23.3.1 zrLV_RELAX5_P9_A

$$\begin{aligned} h_z &\neq h_r \\ g_{9z-} &\neq 0 \\ g_{9r+} &\neq 0 \end{aligned}$$

$$\begin{aligned} V_9 = \frac{h_z^2(2p-1)}{(2p-1)h_z^2 + 2ph_r^2}V_6 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2}V_8 + \\ \frac{(2p+1)g_{9r+}h_z^2h_r + 2pg_{9z+}h_zh_r^2}{(2p-1)h_z^2 + 2ph_r^2} \end{aligned} \tag{23.13}$$

23.3.2 zrLV_RELAX5_P9_B

$$\begin{aligned} h_z &\neq h_r \\ g_{9z-} &= 0 \\ g_{9r+} &= 0 \end{aligned}$$

$$V_9 = \frac{h_z^2(2p-1)}{(2p-1)h_z^2 + 2ph_r^2}V_6 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2}V_8 \tag{23.14}$$

23.3.3 zrLV_RELAX5_P9_C

$$
\begin{aligned}
h_z = h_r &= h \\
g_{9z-} &\neq 0 \\
g_{9r+} &\neq 0
\end{aligned}
$$

$$V_9 = \frac{2p-1}{4p-1}V_6 + \frac{2p}{4p-1}V_8 + \frac{h((2p+1)g_{9r+} + g_{9z+})}{4p-1} \tag{23.15}$$

23.3.4 zrLV_RELAX5_P9_D

$$
\begin{aligned}
h_z = h_r &= h \\
g_{9z-} &= 0 \\
g_{9r+} &= 0
\end{aligned}
$$

$$V_9 = \frac{2p-1}{4p-1}V_6 + \frac{2p}{4p-1}V_8 \tag{23.16}$$

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